# Exercises for Advanced Particle Physics - Winter term 2013/14
## Exercise sheet No. II

Prof. Karl Jakobs, Dr. Romain Madar, Claudia Giuliani

*The solutions have to be returned to mail box no. 1*
*in the foyer of the Gustav-Mie-House before* **Monday, November 11th, 12:00h**.

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## Neutrino oscillations (10 points)

On the last exercise sheet, we studied in details two-flavour neutrino oscillations in the vacuum. We propose to analyze two more general cases, necessary to explain experimental observations: the three-flavours oscillations scheme and the oscillations inside a medium.

**Exercise No. 1 : Three flavours oscillations** **(4 points)**

Since there are three fermion generations, the complete neutrino oscillations picture can be only described in this scheme. We recall that the three flavour neutrino mixing can be described by three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$ and three masses $m_i$ $(i \equiv 1, 2, 3)$.

1. How can muon neutrinos be produced (a description of relevant reactions is expected) ? What is the expected ratio between the different neutrino flavours, if several are produced ?

2. We define the oscillation length due to the mixing between the mass eigenstate $i$ and $j$ by :

$$L_{ij} = \frac{4\pi E}{|\Delta m^2_{ij}|}\ , \quad \Delta m^2_{ij} \equiv m^2_i - m^2_j \tag{1}$$

   Recall why there are only two independent oscillation lengths ? Compute their numerical values for $E = 1$ GeV (we give $|\Delta m^2_{12}| = 7.1\,10^{-5}$ eV$^2$ and $|\Delta m^2_{23}| = 2.5\,10^{-3}$ eV$^2$).

3. By assuming that the oscillations $\nu_e \leftrightarrows \nu_\mu$ and $\nu_\mu \leftrightarrows \nu_\tau$ are described by the two flavour scheme each,

$$\mathcal{P}_{\alpha\to\alpha}(L) \ = \ 1 - \sin^2(2\theta_{ij}) \sin^2\left(\frac{\pi L}{L_{ij}}\right) \tag{2}$$

   explain the result of the Super-Kamiokande experiment shown in Figure 1 (dependence with the zenith angle $\alpha$ and difference between $\nu_e$ and $\nu_\mu$ behavior). A sketch and relevant orders of magnitude are expected. We give the earth diameter $D = 12800$ km.

4. Extract the numerical value of $\theta_{23}$ using Super-Kamiokande results of Figure 1. Hint : the $\nu_\mu$ energy spectrum is so wide that $\sin^2\left(\frac{\pi L}{L_{ij}}\right)$ has to be averaged ($L \ll L_{ij}$ and $L \gtrsim L_{ij}$ cases have to be considered separately).

5. Compute the numerical value of $L_{13}$ for $E = 1$ GeV. Assuming that $\nu_e \leftrightarrows \nu_\tau$ oscillations follow the equation (2), what can we say about the mixing angle $\theta_{13}$ using the Super-Kamiokande results ? Compute the numerical value of $L_{13}$ for reactor neutrinos ($E = 4$ MeV). How far from a nuclear reactor would you put a detector to measurement $\theta_{13}$ ?

6. The Daya Bay collaboration (reactor neutrinos experiment in China) recently measured an (anti-)neutrino survival probability of $0.940 \pm 0.011(\text{stat}) \pm 0.004(\text{syst})$ at a distance $L = 1148$ m. Extract the numerical value of $\theta_{13}$ and its uncertainty. What is the experimental feature of reactor neutrino experiments that allows to reach small systematic uncertainties ?

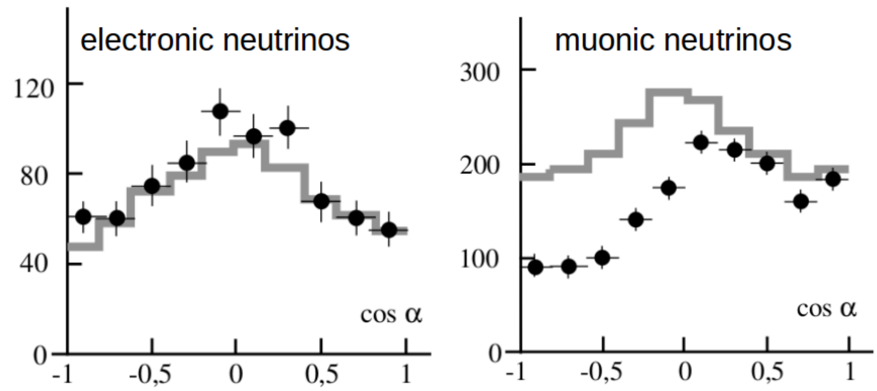


Figure 1: Results from the Super-Kamiokande experiment : number of electronic (left) and muonic (right) neutrinos versus the zenith angle $\alpha$. The black points represent the observed data and the gray line is the prediction without neutrino oscillations.

**Exercise No. 2 : Oscillations in a medium (MSW effect) (4 points)**

The interpretation of the results of solar neutrino experiments cannot be made without a proper treatment of the neutrino propagation in a medium. Let's consider a two flavour oscillation scheme

$$|\nu_e\rangle \;=\; \cos\theta\,|\nu_1\rangle \;+\; \sin\theta\,|\nu_2\rangle \quad ; \quad |\nu_\mu\rangle \;=\; -\sin\theta\,|\nu_1\rangle \;+\; \cos\theta\,|\nu_2\rangle \tag{3}$$

where $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ is the interaction basis and $\{|\nu_1\rangle, |\nu_2\rangle\}$ is the mass basis.

1. By denoting $E_i$ the energy of the state $|\nu_i\rangle$, show that the Hamiltonian

$$\hat{H} = E_1|\nu_1\rangle\langle\nu_1| + E_2|\nu_2\rangle\langle\nu_2| \tag{4}$$

can be written in the $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ basis :

$$\hat{H} = \frac{\Sigma_E}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-\Delta_E\cos 2\theta & \Delta_E\sin 2\theta\\ \Delta_E\sin 2\theta & \Delta_E\cos 2\theta\end{pmatrix} \quad ; \quad \Sigma_E \equiv E_1 + E_2 \;;\; \Delta_E \equiv E_2 - E_1 \tag{5}$$

2. Neutrino interaction are of two kinds : flavour-dependent (only with a lepton of the same flavour as the initial neutrino) and flavour independent. Explain why only the state $|\nu_e\rangle$ will have an additional energy term in a medium.

3. It is possible to show that this additional term is proportional to the electron density $\rho$. By adding a term $\left(\begin{smallmatrix}\alpha\rho & 0\\ 0 & 0\end{smallmatrix}\right)$ to $\hat{H}$, show that we can define an effective mixing angle $\theta_{\mathrm{M}}$ taking into account medium propagation effect, defined by :

$$\tan 2\theta_{\mathrm{M}} = \frac{\Delta_E\sin 2\theta}{\Delta_E\cos 2\theta - \alpha\rho} \tag{6}$$

Hint : we can write

$$\begin{pmatrix}\alpha\rho & 0\\ 0 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\alpha\rho & 0\\ 0 & \alpha\rho\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\alpha\rho & 0\\ 0 & -\alpha\rho\end{pmatrix} \tag{7}$$

4. If a neutrino is created inside a medium, for instance the sun, its evolution will be driven by $\theta_{\mathrm{M}}$ inside the medium. We denote $\{|\nu_{1,\mathrm{M}}\rangle, |\nu_{2,\mathrm{M}}\rangle\}$ the Hamiltonian eigenstate basis in the medium. After having been through the sun, $|\nu_{i,\mathrm{M}}\rangle = \exp i\phi_i\,|\nu_i\rangle$, leading to the following state vector outside the sun :

$$|\psi(t)\rangle = \cos\theta_{\mathrm{M}}\,|\nu_1\rangle\,e^{i\phi_1 - iE_1 t} + \sin\theta_{\mathrm{M}}\,|\nu_2\rangle\,e^{i\phi_2 - iE_2 t} \tag{8}$$

By averaging the oscillating terms (due to energy spectrum average), show that the survival probability of an electronic neutrino on earth is :

$$\langle \mathcal{P}^{\mathrm{M}}_{\nu_e \to \nu_e} \rangle = \frac{1}{2} \left(1 + \cos 2\theta_{\mathrm{M}} \cos 2\theta\right) \tag{9}$$

Plot $\cos 2\theta_{\mathrm{M}}$ versus the electronic density $\rho$. Comment and compare the result to the vacuum neutrino oscillations :

$$\langle \mathcal{P}_{\nu_e \to \nu_e} \rangle = \frac{1}{2} \left(1 + \cos^2 2\theta\right) \tag{10}$$

**Exercise No. 3 : General and opening questions** **(2 points)**

Neutrino oscillations are important because they might have consequences in already well known sectors or might even impact other physics scales.

1. Discuss implications of neutrino mixing for the charged lepton sector ? Which reaction would you study in order to detect these effects ?
2. What would be the consequence of a possible Majorana nature for neutrinos ? Deduce which reaction could be studied to probe the Majorana versus Dirac nature of neutrinos ? Explain why. Hint : you would need two same flavour neutrino in the final state.
3. Is there any possible implications of neutrino masses at non-microscopic scale ?
4. Do you know implications of a non-zero $\theta_{13}$ mixing angle ?