Exercises for Advanced Particle Physics - Winter term 2013/14
Exercise sheet No. III

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The solutions have to be returned to mail box no. 1
in the foyer of the Gustav-Mie-House before **Monday, November 18th, 12:00h**.

Spin-1/2 particle description : the Dirac equation (10 points)

The simple reasoning to obtain the Schrödinger equation from classical mechanics leads to some paradoxes when extrapolated to a relativistic regime. A better desciption of relativistic electrons is given by the Dirac equation and these exercises try to illustrate a few of its important properties.

Exercise No. 1 : Free solutions of the Dirac equation (3 points)

For any linear equation, every motion can be decomposed as a linear combination of plane waves. The set of solution for any plane wave provide a complete basis of every possible motions for the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{1}$$

where x denotes a space-time point $(t, \vec{x})$, ψ is a four-components object (made of two spinors[1]) and γ_μ are (4×4) matrices defined by (I denotes the unity (2×2) matrix, and σ_i are the Pauli matrix[2]):

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ , \ \ \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \ , \ \ i \equiv 1, 2, 3 \tag{2}$$

1. Compute the Dirac equation solution for an electron at rest. Interpret the four solutions.
2. Using the Lorentz transformation of a bi-spinor under a boost $\vec{v}$ along the x_1-axis

$$\psi' = S\psi \ , \ \ S \ = \ \left(\sqrt{\frac{\gamma+1}{2}}\, I_{4\times 4} \ - \mathrm{sign}(\vec{v}\cdot\vec{x}_1)\, \sqrt{\frac{\gamma-1}{2}}\, \gamma_0\gamma_1 \right) \ , \ \ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3}$$

   compute the solution for an electron with momentum $\vec{p} = (p, 0, 0)$. Is it consitent with the general solution presented in the lecture? Hint: you can start with the solution of an electron at rest, obtained at the previous question.

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[1] A spinor is a two-component object which “behaves well” under the rotation, *ie.* respecting the properties of the rotation group. We talk about a *group representation* of dimension 2 (1D=scalar, 3D=vector, etc ...)

[2] These matrices describes how a rotation will change a spinor. Their properties are unrelated to their (2×2) size, but rather to the *rotation group itself.* For example, 3×3 rotation matrices have the exact same properties.

Exercise No. 2 : Few invariants built on Dirac bi-spinors **(2 points)**

Finding invariant quantities under simple transformations is crucial to build meaningful theories. If a phenomenon is invariant under rotation, a proper equation cannot include only r_x but must involve (r_x, r_y, r_z) in a well defined combination (respecting rotations). This exercise discusses two important invariants which can be constructed out of the four-component object ψ.

1. Show that the number $\psi^\dagger\psi$ is not invariant under a Lorentz transformation (use equation (3)). However, is the number $\bar{\psi}\psi$ invariant, where $\bar{\psi} \equiv \psi^\dagger\gamma_0$?

2. Using the general property of S (defined by $\psi' = S\psi$)

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu, \tag{4}$$

   show that the quantiy $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ is a 4-vector.

Exercise No. 3 : Non relativistic limit of the Dirac equation **(5 points)**

Consider this form of the Dirac Equation:

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\psi = i\partial_t\psi, \tag{5}$$

and a non-relativistic electron travelling at speed $v \ll 1$ represented by $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ where ψ_1 and ψ_2 are spinors.

1. If there is an electromagnetic field $A^\mu = (A^0, \vec{A})$, the dynamic of an electron is described by the Dirac equation where $\vec{p} \to \vec{p}+e\vec{A}$ and $E \to E+eA^0$. By searching for stationnary solutions $\psi = e^{-iEt}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$, show that ψ_A fulfills

$$\left(\frac{1}{2m}\left(\vec{\sigma}\cdot(\vec{p}+e\vec{A})\right)^2 - eA^0\right)\psi_A = E_{\text{kin}}\psi_A \tag{6}$$

   where E_{kin} is the kinetic energy of the electron. What can you say about ψ_B? Discuss the result. Hints: why can you assume $|eA^0| \ll m$ and $E_{\text{kin}} \ll m$?

2. Using $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\partial_t\vec{A} - \vec{\nabla}A^0$, show that ψ_A fulfills the Pauli equation:

$$\left(\frac{1}{2m}(\vec{p}+e\vec{A})^2 + \frac{e}{2m}\vec{\sigma}\cdot\vec{B} - eA^0\right)\psi_A = E_{\text{kin}}\psi_A, \tag{7}$$

   and derive the gyromagnetic ratio g (or Landé g-factor) of the electron, definey by:

$$\vec{\mu} \equiv -g\frac{e}{2m}\vec{S}\ , \quad \vec{S} \equiv \frac{1}{2}\vec{\sigma} \tag{8}$$

   Hint: You can make use of the following:

   - $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$, in the case where $[\vec{a},\vec{\sigma}] = [\vec{b},\vec{\sigma}] = 0$
   - $\vec{\nabla}\times(\vec{A}\psi) + \vec{A}\times(\vec{\nabla}\psi) = (\vec{\nabla}\times\vec{A})\psi$.

3. What is the classical prediction of the gyromagnetic ratio g for an orbital angular momentum? Do you know few experiments where this Landé factor is essential to explain observations?

4. *Bonus.* Do you know experiments where the Landé factor measurement is inconsistent with the Dirac equation prediction? Do you know if this is theoretically understood ?