

Exercises for Advanced Particle Physics - Winter term 2013/14

Exercise sheet No. IV

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*The solutions have to be returned to mail box no. 1
in the foyer of the Gustav-Mie-House before Monday, November 25th, 12:00h.*

Elementary processes, transition amplitudes and cross sections

During the previous lectures, the propagation of free particles was described in a relativistic framework. To better understand particle physics, we want to learn about how particles *interact*. Experimentally, we analyse the final state of collisions produced with a known initial state. These exercises aim to illustrate how the properties of the final state can be predicted by a given theory.

Exercise No. 1: Generic form of the cross section (1+2 points)

Based on Fermi's Golden Rule, the generic form of the cross section can be written, as seen in the lecture:

$$d\sigma = \frac{1}{F} |\mathcal{M}|^2 dQ \quad (1)$$

1. Discuss the physical meaning of each term in this equation.
2. *Optional question (2 points)*. For the reaction $a + b \rightarrow c + d$ with a total energy E_{cm} , show that the differential cross section in the centre-of-mass frame ($\mathbf{p}_a + \mathbf{p}_b = \mathbf{0}$) can be written

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2} \frac{|\mathbf{p}_c|}{|\mathbf{p}_a|} \quad (2)$$

Hint: Use the following relations:

$$F = 4 \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2} \quad (3)$$

$$dQ = \frac{d\mathbf{p}_c}{(2\pi)^3 2E_c} \frac{d\mathbf{p}_d}{(2\pi)^3 2E_d} (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d) \quad (4)$$

and the following transformation of the integration variable

$$p \rightarrow \sqrt{m_c^2 + p^2} + \sqrt{m_d^2 + p^2} \quad (5)$$

Exercise No. 2: Creation of muon pairs in e^+e^- collisions in QED (9 points)

The goal of this exercise is to calculate and interpret the unpolarized differential and total cross section of the following process predicted by quantum electrodynamics.

$$e^+(p_1) e^-(p_2) \rightarrow \mu^+(p_3) \mu^-(p_4) \quad (6)$$

(p_1, p_2, p_3 and p_4 denote the four momenta of the particles).

1. Draw the Feynman diagram of the process. Following the structure "current \times propagator \times current", write the associated invariant amplitude $i \cdot \mathcal{M}$ (note $q \equiv p_2 + p_1$).

2. Show that the squared amplitude is given by

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} [\bar{v}(p_1)\gamma^\mu u(p_2)\bar{u}(p_2)\gamma^\nu v(p_1)] [\bar{u}(p_4)\gamma_\mu v(p_3)\bar{v}(p_3)\gamma_\nu u(p_4)] \quad (7)$$

3. We assume that the experimental setup is neither able to produce polarized beams nor able to measure the spin state of the out-coming particles. Explain why the expression of the matrix element involved in equation (1) is

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \quad (8)$$

4. Show that

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4q^4} \text{tr} \left[(\not{p}_1 - m_e)\gamma^\mu (\not{p}_2 + m_e)\gamma^\nu \right] \text{tr} \left[(\not{p}_4 + m_\mu)\gamma_\mu (\not{p}_3 - m_\mu)\gamma_\nu \right] \quad (9)$$

Hint: Write in spinor indices, and use the completeness relations $\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m$ and $\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m$.

5. Using the approximation $m_e \rightarrow 0$ and the theorems for traces involving γ matrices from the lecture, show that this expression can be simplified to

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4} [(p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_3)(p_1 \cdot p_4) + m_\mu^2(p_1 \cdot p_2)] \quad (10)$$

6. Considering that the e^+e^- collision are symmetric ($\mathbf{p}_1 = -\mathbf{p}_2$), show that $\overline{|\mathcal{M}|^2}$ can be simplified to

$$\overline{|\mathcal{M}|^2} = e^4 \left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right] \quad (11)$$

where θ is the angle between p_1 and p_3 , E is the energy of any of the four fermions.

7. Using equation (2), show that the differential cross section and the total cross section are given by ($\alpha \equiv e^2/(4\pi)$)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right] \quad (12)$$

$$\sigma_{\text{total}} = \frac{\pi\alpha^2}{3E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{m_\mu^2}{2E^2} \right) \quad (13)$$

8. Discussion of the result.

- What minimum energy of the incoming fermions is required for this result to be valid?
- What would be $\sigma(E)$ if \mathcal{M} were constant? Compare with the behaviour of $\sigma(E)$ defined in equation (13) and deduce an experimental way to probe QED interactions.
- Express these cross sections in terms of the centre-of-mass energy \sqrt{s} using the high-energy approximation $E \gg m_\mu$. Interpret the result.